Stabilizer complexity of quantum states and connections with frame theory

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Based on joint work with Mehrdad Tahmasbi

AMS meeting 2026

Motivation

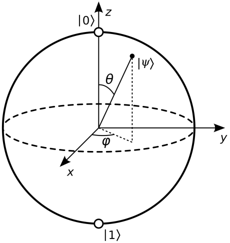
- Stabilizer states are central in quantum information science (error correction, simulation, etc).
- **This talk:** "View stabilizer states as **mathematical frames** and study the **sparsest decomposition** of quantum states into these frames."
- We mainly focus on three measures:

  stabilizer rank, the **Gowers norm** and **stabilizer fidelity**.

Quantum states

- **Qubit:** The state of a two level-system quantum is specified by $|\psi\rangle \in \mathbb{C}^2$
- $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, $\alpha, \beta \in \mathbb{C}$ are called amplitudes $|\alpha|^2 + |\beta|^2 = 1$. Upon measurement

$$Pr(0) = |\alpha|^2, \quad Pr(1) = |\beta|^2.$$



- **General quantum states:** $|\psi\rangle \in \mathbb{C}^d$ for $1 \leq d \leq \infty$, $|\psi\rangle = \sum_j \alpha_j\,|j\rangle$ with $\sum_j |\alpha_j|^2 = 1$. Upon measurement, we obtain

$$Pr(j) = |\alpha_j|^2.$$

Quantum operations

- $U \in \mathbb{C}^{d\times d}$ is a **unitary matrix** if $U^\dagger = U^{-1}$ where $U^\dagger_{jk} = U^*_{kj}$. It is a **generalization of orthogonal matrices** to the complex number field.
- The evolution of a closed system is according to a **unitary matrix** (which preserves the law of probability).

$$U : |\psi\rangle \in \mathbb{C}^d \mapsto U|\psi\rangle$$

- Basic unitary **gates**:

Pauli: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

Clifford: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

Non-Clifford: $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.

Pauli Matrices, Pauli Group, and Clifford Group

Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Together with the identity I, they form the single-qubit Pauli operators.
***n*-qubit Pauli group** $\mathcal{P}_n$**:**

$$\mathcal{P}_n = \left\{ i^k \, P_1 \otimes \cdots \otimes P_n \;\middle|\; k \in \{0,1,2,3\},\; P_j \in \{I, X, Y, Z\} \right\}$$

- Closed under multiplication (up to phases $\{\pm 1, \pm i\}$)
- Elements either commute or anticommute

Clifford group Cliff_n**:**

$$\mathrm{Cliff}_n = \left\{ U \in U(2^n) \;\middle|\; UPU^\dagger \in \mathcal{P}_n \;\; \forall P \in \mathcal{P}_n \right\}$$

- Normalizer of the Pauli group
- Generated by $\{H, S, \mathrm{CNOT}\}$
- Maps Pauli operators to Pauli operators under conjugation

Background: Three definitions for stabilizer states

1. **Definition via abelian subgroup of Pauli group:** $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state iff there exists an abelian subgroup of $\mathcal{P}$ with 2^n elements such that for any $P \in \mathcal{P}_n$, $P|\phi\rangle = |\phi\rangle$.
2. **Definition via quadratic phases:** $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is a stabilizer state iff there exists an affine subspace $A \subseteq \mathbb{F}_2^n$, and quadratic Q and linear ℓ functions : $\mathbb{F}_2^n \to \mathbb{F}_2$ such that:

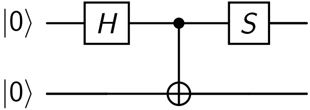
$$|\phi\rangle = \frac{1}{\sqrt{|A|}} \sum_{x \in A} (-1)^{Q(x)} i^{\ell(x)} |x\rangle$$

3. **Definition via invariant under Clifford operations:** The set of stabilizer states is the orbit of $|0^n\rangle$ under Cliff $= \langle H, S, CNOT \rangle$.
   That means each stabilizer state can be produced by a Clifford circuit starting with $|0^n\rangle$.

Example of stabilizer states

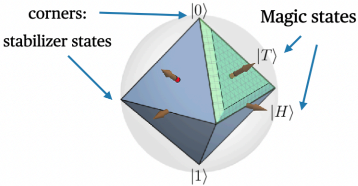
For instance consider the quantum state $\frac{|00\rangle+i|11\rangle}{\sqrt{2}}$

1. It is stabilized by Z_1Z_2 and X_1Y_2.
2. $A=\{00,11\}$, $Q(x)=1$, $\ell(x)=x_1$
3. Here is the Clifford circuit



The T state

$$|T\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$



corners:

stabilizer states

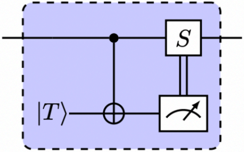
Magic states

credit: Dawkins Howard, PRL

Example: 1 qubit

Magic state teleportation:

Clifford circuits on specific “magic” sates can simulate universal quantum computations.



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Measures of Stabilizer Complexity

- **Stabilizer fidelity** $F(\psi)$ of a quantum state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is the maximum overlap between $|\psi\rangle$ and any stabilizer state.

$$F(\psi) = \max_{|s\rangle \in \mathrm{Stab}_n} |\langle \psi | s \rangle|$$

- **Approximate stabilizer rank** $\chi_\delta(\phi)$ of a quantum state $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ is the smallest number r such that there exists $c_1, \ldots, c_r \in \mathbb{C}$ and $|s_1\rangle, \ldots, |s_r\rangle \in \mathrm{Stab}_n$ such that

$$\| \, |\phi\rangle - (c_1 |s_1\rangle + \ldots + c_r |s_r\rangle) \| \leq \delta.$$

- Measure **based on the Gowers norm** which we will define shortly

Why Stabilizer Rank Matters

Stabilizer rank:

- Stabilizer rank measures how “far” a state is from the efficiently simulable stabilizer world

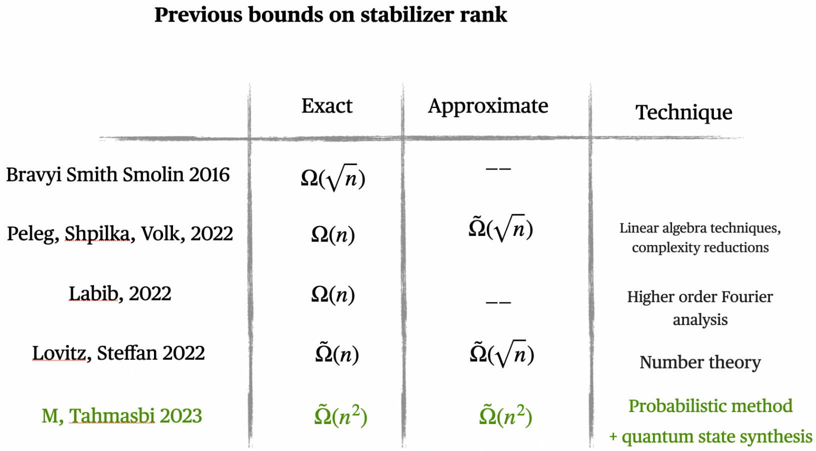
Why is it important?

- **Classical simulation:**
  - Stabilizer states ⇒ efficient simulation (Gottesman–Knill)
  - If $|\psi\rangle$ has stabilizer rank r, then many quantities can be simulated in time $\text{poly}(n) \cdot r$
- **Resource theory of magic:**
  - Stabilizer rank quantifies non-Clifford resources (“magic”)
  - Low rank ⇒ weak quantum advantage
- **Bridging classical and quantum complexity:**
  - Small stabilizer rank ⇒ efficient classical approximation algorithms
  - Large stabilizer rank believed necessary for quantum speedups

Takeaway:

Stabilizer rank is a **complexity measure** controlling the hardness of simulating quantum systems.

Main result 1



Previous bounds on stabilizer rank

	Exact	Approximate	Technique
Bravyi Smith Smolin 2016	$\Omega(\sqrt{n})$	——	
Peleg, Shpilka, Volk, 2022	$\Omega(n)$	$\tilde{\Omega}(\sqrt{n})$	Linear algebra techniques, complexity reductions
Labib, 2022	$\Omega(n)$	——	Higher order Fourier analysis
Lovitz, Steffan 2022	$\tilde{\Omega}(n)$	$\tilde{\Omega}(\sqrt{n})$	Number theory
M, Tahmasbi 2023	$\tilde{\Omega}(n^2)$	$\tilde{\Omega}(n^2)$	Probabilistic method + quantum state synthesis

Major open question

$P \neq NP$?

Can we show that NP-complete problems do not have short representation within a specific model? (e.g. circuits with specific structure, …)

Specific model: linear combination of an overcomplete functional basis.
In particular quadratic phases

Williams CCC 2018: For any k there is a function $f : \{0,1\}^n \to \{0,1\}$ in NP such that that in any decomposition $f(x) = \sum_{i=1}^{r} c_i(-1)^{Q_i(x)}$ into quadratic phases $r \geq n^k$.

Open question:
Can we prove the same thing for functions in P?
Can give an example of a function in P which requires $r = \omega(n)$ representation?

M, Tahmasbi 2023: an example of a function that requires $\tilde{\Omega}(n^2)$ terms

Open question:
Quadratic uncertainty principle

Show that the AND i.e. $(-1)^{x_1 \cdots x_n}$ function requires exponential representation into quadratic phases

Second application: Quantum Property Testing

Let $\rho \in \mathbb{C}^{d\times d}$.

- **Full-state tomography:** Given $\rho^{\otimes n}$ output an approximation to ρ.
- **Sample complexity:** We need $n = \Omega(d^2/\epsilon^2)$ samples to estimate ρ within ϵ trace distance [Haa+16].
- If $\rho \in (\mathbb{C}^{2\times 2})^{\otimes m}$, $d = 2^m$; we need exponentially many samples in m.
- **Quantum property testing:** Testing specific properties of a quantum state using much fewer samples
- E.g., testing if the rank of a density matrix is a given constant requires only a constant number of copies [OW15]
- **Main application 2:** Property testing for stabilizerness

Tolerant Testing of Stabilizer States

Problem Statement

Given a quantum state $|\psi\rangle$ with the promise that its stabilizer fidelity $F(\psi)$ satisfies:

$$F(\psi) \geq \epsilon_1 \quad \text{or} \quad F(\psi) \leq \epsilon_2,$$

decide which case holds.

- This generalizes exact testing by allowing a **gap** between acceptance and rejection thresholds.
- Known as *tolerant property testing* in theoretical computer science.

Existing Results for Testing Stabilizer States

Exact Testing ($\epsilon_1 = 1$)

[Mon17] $\mathcal{O}(n)$ copies suffice to *identify* an n-qubit stabilizer via Bell sampling.

[GNW21] Only **6 copies** needed to test exact stabilizer states (vs. far in fidelity) using Bell difference sampling (via a variant of Schur–Weyl duality for Cliff gp).

Tolerant Testing ($\epsilon_1 < 1$)

[Gre+22] $\mathcal{O}(k^{12})$ copies sufficient to distinguishing between $\epsilon_1 \geq 1/k$ vs. $\leq 2^{-O(n)}$ (e.g. Haar-random ensemble).

[Gre+24b] showed that finding a stabilizer state with good overlap with a given state using polynomial samples and exponential post-proc. (improved to poly-time in [Che+24])

[Gre+24a] For $\epsilon_2 \leq \frac{4\epsilon_1^6 - 1}{3}$ and $O(\text{poly}(1/\epsilon_1))$ samples (limits $\epsilon_1 \geq \sqrt[3]{1/2}$).

[AD24] Improved to work with $\epsilon_2 \leq 2^{-\text{poly}(1/\epsilon_1)}$, for phase states assuming $\epsilon_2 = \epsilon^{\mathcal{O}(1)}$ (Complexity: $\text{poly}(n/\epsilon_1)$)

Main result 2: Tolerant Testing

Theorem (Improved bounds on tolerant testing)

Let $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ *with the promise that either* $F(\phi) \geq \epsilon_1$ *or* $\leq \epsilon_2$. *Using poly*$(1/\epsilon_1)$ *copies of a quantum state* $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ *and a circuit of size* $n \cdot \text{poly}(1/\epsilon_1)$ *we can distinguish between the two cases with probability of error* $\leq 1/3$ *provided that* $\epsilon_2 \leq \epsilon_1^C$ *for a sufficiently large absolute constant* $C > 0$.

Tool: Gowers-3 norm

- For a function $f : \mathbb{F}_2^n \to \mathbb{C}$ define Gowers 3-norm:

$$\|f\|_{U^3}^8 = \frac{1}{16^n} \sum_{x,h_1,h_2,h_3 \in \mathbb{F}_2^n} f(x)\overline{f(x+h_1)f(x+h_2)f(x+h_3)} \times f(x+h_1+h_2)f(x+h_1+h_3)f(x+h_2+h_3)\overline{f(x+h_1+h_2+h_3)}$$

Theorem (see [HHL19])

- If P is a polynomial of degree 2 then $\|(-1)^P\|_{U^3} = 1$.
- For any polynomial $P : \mathbb{F}_2^n \to \mathbb{F}_2$ of degree 2 it holds that $\|f\|_{U^3} \leq |\langle (-1)^P, f\rangle|$.
- For any polynomial $P : \mathbb{F}_2^n \to \mathbb{F}_2$ of degree 2, if $\|f\|_{U^3} \geq \epsilon$, then $|\langle (-1)^P, f\rangle| \geq \delta(\epsilon, P)$.

For $f, g : \mathbb{F}_2^n \to \mathbb{C}$, $\langle f, g\rangle := \mathbb{E}(f(x)\bar{g}(x))$.

Main Theorem: Gowers Norm & Fidelity

Definition (quantum Gowers norm)

For a quantum state $|\psi\rangle = \frac{1}{\sqrt{N}} \sum_x g(x) |x\rangle$, let $\| \, |\phi\rangle \, \|_{U^3} := \|g\|_{U^3}$.

Theorem

If $\| \, |\phi\rangle \, \|_{U^3}^8 \geq \gamma$, then $\exists$ stabilizer $|s\rangle$ such that $|\langle \phi | s \rangle| \geq \gamma^{C_2}/C_1$ for some absolute constants C_1, C_2.

Theorem

[AD '24] There is a quantum algorithm that uses $O(1/\delta)$ copies of a quantum state $|\psi\rangle$ and outputs a δ additive estimation to a number R such that

$$\| \, |\psi\rangle \, \|_{U^3}^{16} \leq R \leq \| \, |\psi\rangle \, \|_{U^3}^{8}.$$

Zauner's Conjecture and Stabilizer Complexity

SIC-POVMs and fiducial states. A set of unit vectors $\{|\psi_i\rangle\}_{i=1}^{d^2} \subset \mathbb{C}^d$ forms a *symmetric informationally complete POVM* (SIC-POVM) if

$$|\langle\psi_i, \psi_j\rangle|^2 = \frac{1}{d+1} \qquad \text{for all } i \neq j.$$

A *fiducial state* is a single state $|\phi\rangle$ whose orbit under the Weyl–Heisenberg group generates such a SIC.

Zauner's conjecture. For every dimension d, there exists a Weyl–Heisenberg SIC-POVM, equivalently a fiducial state whose orbit gives a SIC.

Why are fiducial states interesting here?

- They are in a sense maximally spread out under the Pauli/Weyl action.
- Their overlaps with the Weyl orbit are extremely uniform.
- Overlap between stabilizer states, on the other hand, are far from uniform
- This suggests they may be very far from the highly algebraic stabilizer world.

Question.

What is the stabilizer complexity of SIC fiducial states?

Speculation.

Fiducial states are likely *extremally far* from the stabilizer set,
perhaps having very small stabilizer fidelity and very large stabilizer rank.

Open questions

We end with some open questions

- Tolerant testing of stabilizer rank?
- Tight bounds relating Gowers norms and stabilizer fidelity?
- Extremal examples (e.g., SIC fiducial states)?

Thank you for your attention!

Generalized Pauli operators

- **Generalized Pauli operators:**

  $$X^a = X^{a_1} \otimes \ldots \otimes X^{a_n}, Z^b = Z^{b_1} \otimes \ldots \otimes Z^{b_n}$$

  with $a, b \in \mathbb{F}_2^n$. They satisfy

  $$X^a \left|x\right\rangle = \left|x+a\right\rangle, \quad Z^b \left|x\right\rangle = (-1)^{\langle x,b\rangle} \left|x\right\rangle,$$

  where $\langle x, b\rangle = \sum_i x_i b_i$ (additions $\mod 2$).
- **Weyl operator:** $W(z) := i^{\langle y,\alpha\rangle} X^y Z^\alpha$, $z = (y, \alpha) \in \mathbb{F}_2^{2n}$.
- **Symplectic inner product:** Let $z_1 = (y_1, \alpha_1), z_2 = (y_2, \alpha_2) \in \mathbb{F}_2^{2n}$,

  $$[z_1, z_2] := \langle y_1, \alpha_2\rangle + \langle y_2, \alpha_1\rangle .$$

- **Commutation relationships:**

  $$W(z_1)W(z_2) = (-1)^{[z_1,z_2]} W(z_2)W(z_1)$$

- $T \subset \mathbb{F}_2^{2n}$ **Lagrangian subspace** if $|T| = 2^n$ and for any $z_1, z_2 \in T$, $[z_1, z_2] = 0$ (they commute). T corresponds to a stabilizer group.

Characteristic function

Definition (Characteristic function)

For $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ define the characteristic function $f_\phi : \mathbb{F}_2^{2n} \to \mathbb{R}$ with $f_\phi(z) = |\langle\phi| W(z) |\phi\rangle|^2$.

Properties:

- **Normalization:**

$$\frac{1}{N} \sum_{z \in \mathbb{F}_2^{2n}} f(z) = 1$$

  furthermore $f(0) = 1$.

Lemma (Fidelity and characteristic function; [GNW21]; [Gre+24b])

Let $|\phi\rangle \in (\mathbb{C}^2)^{\otimes n}$ *and a Lagrangian subspace* $T \subset \mathbb{F}_2^{2n}$ *then* $F(\phi) \geq \sum_{z \in T} f_\phi(z)$.

Lemma

If $V \subseteq \mathbb{F}_2^{2n}$ *is a subspace, then for any* $z_0 \in \mathbb{F}_2^{2n}$, $\sum_{z \in V} f(z) \geq \sum_{z \in V} f(z + z_0)$.

A corollary of these lemmas is that if we can find $V \subset \mathbb{F}_2^{2n}$ such that $\sum_{z \in V} f(z) \geq \epsilon$, and furthermore V can be covered by K affine subspaces then $F(\phi) \geq \epsilon / K$.

Proof outline for phase states

We first outline how [AD24] achieve this bound for phase states[1]. Proof follows closely that of the classical inverse Gowers theorem[2]. It consists of two major steps.

1. Probabilistic construction of an almost linear function,
2. Use this map to find an almost linear subspace with high weight
3. Use additive combinatorics to cover this subspace with a few affine subspaces
4. Find a Lagrangian subspace with high weight $\implies$ lower bound on fidelity

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[1] Taking the form $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_x (-1)^{f(x)} |x\rangle$

[2] Weaker quasipolynomial bound were mentioned in [MT24].

Probabilistic Construction of an almost linear function

Construction: Probabilistically construct $\zeta : \mathbb{F}_2^n \to \mathbb{F}_2^n$ via:

- Choose $\zeta(y) = \alpha$ with probability $f(y, \alpha)$.

Properties:

Assuming $\|\phi\|_{U^3} \geq \gamma$, this function has the following properties:

- $f(y, \zeta(y)) = \text{poly}(\gamma)$ for poly(γ) fraction of y.
- $\zeta(x + y) = \zeta(x) + \zeta(y)$ for poly(γ) fraction of x, y.

Limitation: ζ is a valid probability distribution only for phase states.
For instance if $|\phi\rangle = |0\rangle$, then $f_\phi(y, \alpha) = 0$ for all $y \neq 0$.

Tools from additive combinatorics (overview)

- Using the almost-linear ζ we find an almost linear subspace with high weight.
- Using Balog-Szemerédi-Gowers [BS94] we find a subset S with small doubling: $|S+S| \leq K|S|$.
- Apply resolution to Marton's conjecture ([Gow+23]) to cover S using a few translates of a linear subspace.
- Cover this subspace with a few Lagrangians.

Conclusion:

- Find linear map ℓ such that $\mathbb{E}_y|\langle\phi| X^y Z^{\ell(y)} |\phi\rangle|^2 = \mathrm{poly}(\gamma)$.
- From this, recover a stabilizer "phase" state with overlap poly(γ).

Tools from additive combinatorics (details)

Theorem (Balog and Szemerédi [BS94])

Let $(G, +)$ *be an Abelian group and* $S \subset G$ *with* $\Pr_{z_1, z_2 \in S}[z_1 + z_2 \in S] \geq \epsilon$. *There exists,* $S' \subset S$ *with* $|S'| \geq \frac{\epsilon |S|}{3}$ *and* $|S' + S'| \leq (\frac{6}{\epsilon})^8 |S|$.

Theorem ([Gow+23])

Let $S \subset \mathbb{F}_2^n$ *be a subset with* $|S + S| \leq K|S|$. *Then,* S *can be covered with* $(2K)^8$ *transitions of a subspace* $V \subset \mathbb{F}_2^n$ *with* $|V| \leq |S|$

Combining the above theorems yields the following corollary, which is the main tool we need from additive combinatorics.

Corollary

Let $S \subset \mathbb{F}_2^n$ *with* $\Pr_{z_1, z_2 \in S}[z_1 + z_2 \in S] \geq \epsilon$. *There exists an affine subspace* $V \subset \mathbb{F}_2^n$ *with* $|S \cap V| \geq \frac{\epsilon^{K_2}}{K_1} |S|$ *and* $|V| \leq |S|$ *where* $K_1, K_2 \geq 0$ *are absolute constants. One can choose* $K_2 = 73$ *and* $K_1 = 3 \times 6^{72}$.

Non-Phase States: Limitations of Phase-State Proof

Limitations of the previous approach for non-phase states:

- The stabilizer state produced from the previous approach is a phase state. However if we consider a non-phase state such as the computational basis, overlap with any phase state can be as small as $1/\sqrt{N}$.
- For phase states, for any y, $f(y, \alpha)$ is a probability distribution: $\sum_\alpha f(y, \alpha) = 1$.
- For general states $\sum_\alpha f(y, \alpha)$ can be concentrated on few y.

Ideas used by [AD24]:

- Sample (y, α) using $f(y, \alpha)$, then linearize using additive combinatorics.
- Leads to non-commuting elements: need $\exp(\mathrm{poly}(\gamma^{-1}))$ many stabilizer subspaces.
- Implies a stabilizer state with weak overlap $\geq \exp(-\mathrm{poly}(\gamma^{-1}))$.
- Relied on a conjecture

Our Approach: Random Clifford Trick

Idea: Apply random Clifford C to $|\phi\rangle$.

- We use two properties of random Clifford:
  1. Stabilizer measures invariant under Clifford operations.
  2. Clifford group is a 3-design.
- This balances the quantum state: We show that for any quantum state $|\phi\rangle$ there exist some Clifford C such that $f_{C|\phi\rangle}(y,\alpha)$: $\sum_\alpha f_{C|\phi\rangle}(y,\alpha) = O(1)$ for all y.
- We modify the phase-state proof to work in this case.

Impact:

- Better bounds, no conjectures.

Remark: Around the same time we put our work online (literally the day before), two groups [ABD24; BDH24] achieved similar exponential to polynomial bound. They use original techniques plus graph/algebraic tools to handle non-commuting elements.

Bell Sampling and Gowers Norm Estimation

Given two copies of a state $|\phi\rangle$, *Bell sampling* is the process of measuring in the Bell basis:

$$\left\{|W_z\rangle := (W_z \otimes \mathbb{I})\,|\Phi^+\rangle : z \in \mathbb{F}^{2n}\right\}$$

where $|\Phi^+\rangle := \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{F}^n} |x\rangle\,|x\rangle$ is the maximally entangled state.

Given four copies $|\phi\rangle^{\otimes 4}$, *Bell difference sampling* performs Bell sampling on the first two and last two copies to obtain $z_1, z_2 \in \mathbb{F}^{2n}$, then outputs $z = z_1 + z_2$.

From [GNW21, Eq (3.1)], the probability that Bell measurements on two copies give the same output is:

$$\frac{1}{2}\left(1+\lambda\right), \qquad \text{where } \lambda := \sum_{z\in\mathbb{F}^{2n}} q(z)f(z)$$

where f is the characteristic function of $|\phi\rangle$, $q = f * f$, where $(f * g)(x) = \mathrm{E}_{y\in\mathbb{F}_2^n}[f(y)g(x+y)]$.

Lemma ([AD24], Lemma 3.8)

We can estimate λ up to error $\pm\delta$ using $O(1/\delta^2)$ copies and a circuit of size $O(n/\delta^2)$.

This quantity is related to the Gowers 3-norm as [AD24, Eq. (4)]:

$$\||\phi\rangle\|_{U^3}^{16} \leq \sum_{z\in\mathbb{F}^{2n}} q(z)f(z) \leq \||\phi\rangle\|_{U^3}^{8}$$

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